

B-Fredholm Elements in Rings and Algebras

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Abstract

In this paper, we study B-Fredholm elements in rings and algebras. After characterising these elements in terms of generalized Fredholm elements, we will give sufficient conditions on a unital primitive Banach algebra A , under which we prove that an element of A is a B-Fredholm element of index 0 if and only if it is the sum of a Drazin invertible element of A and an element of the socle of A .

1 Introduction

This paper is a continuation of [5], where we defined B-Fredholm elements in semi-prime Banach algebras, and we focused our attention on the properties of the index. In particular, we gave a trace formula for the index of B-Fredholm operators. Here we will consider in a first step B-Fredholm elements in the case of general rings, and then consider the case of primitive Banach algebras. Let X be a Banach space and let $L(X)$ be the Banach algebra of Bounded linear operators acting on X . In [6], we have introduced the class of linear bounded B-Fredholm operators. If $F_0(X)$ is the ideal of finite rank operators in $L(X)$ and $\pi : L(X) \longrightarrow A$ is the canonical projection, where $A = L(X)/F_0(X)$, it is well known by the Atkinson's theorem [4, Theorem 0.2.2, p.4], that $T \in L(X)$ is a Fredholm operator if and only if its projection $\pi(T)$ in the algebra A is invertible. Similarly, in the following result, we established an Atkinson-type theorem for B-Fredholm operators.

THEOREM 1.1. *[10, Theorem 3.4]: Let $T \in L(X)$. Then T is a B-Fredholm operator if and only if $\pi(T)$ is Drazin invertible in the algebra $L(X)/F_0(X)$.*

Tacking into account this result and the definition of Fredholm elements given in [3], we defined in [5], B-Fredholm elements in a semi-prime Banach algebra A , modulo an ideal J of A .

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DEFINITION 1.2. [5] Let A be unital semi-prime Banach algebra and let J be an ideal of A , and $\pi : A \rightarrow A/J$ be the canonical projection. An element $a \in A$ is called a B-Fredholm element of A modulo the ideal J if its image $\pi(a)$ is Drazin invertible in the quotient algebra A/J .

Recall that a ring A is semi-prime if for $u \in A$, $uxu = 0$, for all $x \in A$ implies that $u = 0$. A Banach algebra A is called semi-prime if A is also a semi-prime ring.

In a recent work [13], Cvetkovic and al., gave in [13, Definiton 2.3] a definition of B-Fredholm elements in Banach algebras. However, their definition does not englobe the class of B-Fredholm operators, since the algebra $L(X)/F_0(X)$ is not a Banach algebra. That's why in our definition, we consider general algebras, not necessarily being Banach algebras, so it includes also the case of the algebra $L(X)/F_0(X)$.

Recall that Fredholm element in a semi-prime ring A were defined in [3], as follows:

DEFINITION 1.3. [3, Definition 2.1] An element $a \in A$ is said to be a Fredholm element of A modulo J if $\pi(a)$ is invertible in the quotient ring A/J , where $\pi : A \rightarrow A/J$ is the canonical projection.

Here we will use Definition 1.3 and Definition 1.2 respectively, to define Fredholm elements and B-Fredholm elements in a unital ring . As we will see in section 2, B-Fredholm elements in a ring, and similarly to B-Fredholm operators acting on a Banach space as observed in [9, Proposition 3.3] , are related to generalized Fredholm operators which had been defined in [12] and studied later in [19]. Thus we will prove in section 2, that an element a of a ring A with a unit e , is a B-Fredholm element of A modulo and ideal J of A if and only if there exists an integer $n \in \mathbb{N}$, an element $c \in A$ such that $a^n c a^n - a^n \in J$ and $e - a^n c - c a^n$ is a Fredholm element in A modulo J . Moreover, we will prove a spectral mapping theorem for the Fredholm spectrum and the B-Fredholm spectrum for elements in a unital Banach algebra.

In section 3, we will be concerned by B-Fredholm elements in a unital primitive Banach algebra A modulo the socle of A . We will give a condition on the socle of A , under which we prove that an element of A is a B-Fredholm element of index 0 if and only if it is the sum of a Drazin invertible element of A and an element of the socle of A , extending a similar decomposition given for B-Fredholm operators acting on a Banach space in [9, Corollar 4.4]. Moreover, if p is any minimal idempotent in A , for $a \in A$, consider the operator $\hat{a} : Ap \rightarrow Ap$, defined on the Banach space Ap , by $\hat{a}(y) = ay$, for all $y \in Ap$. Then, we will

give conditions under which there is an equivalence between a being a B-Fredholm element of A , and \widehat{a} , being a B-Fredholm operator on the Banach space Ap .

2 B-Fredholm elements in ring

Except when it is clearly specified, in all this section A will be ring with a unit e , J an ideal of A , and $\pi : A \longrightarrow A/J$ will be the canonical projection.

DEFINITION 2.1. A non-empty subset \mathbf{R} of A is called a regularity if it satisfies the following conditions:

- If $a \in A$ and $n \geq 1$ is an integer, then $a \in \mathbf{R}$ if and only if $a^n \in \mathbf{R}$,
- If $a, b, c, d \in \mathcal{A}$ are mutually commuting elements satisfying $ac + bd = e$, then $ab \in \mathbf{R}$ if and only if $a, b \in \mathbf{R}$.

Recall also that an element $a \in A$ is said to be Drazin invertible if there exists $b \in A$ and $k \in \mathbb{N}$ such that $bab = b, ab = ba, a^kba = a^k$.

THEOREM 2.2. *The set of Fredholm elements in A modulo J is a regularity.*

Proof. It is well known that the set of invertible elements in the quotient ring A/J is a regularity. Thus, its inverse image by the ring homomorphism π is a regularity.

THEOREM 2.3. *The set of B-Fredholm elements in A modulo J is a regularity.*

Proof. Similarly to [10, Theorem 2.3], where it is proved that the set of Drazin invertible elements in a unital algebra is a regularity, we can prove that in the quotient ring A/J the set of Drazin invertible elements is a regularity. Thus, its inverse image by the homomorphism π is also a regularity.

PROPOSITION 2.4. *Let a_1, a_2 be B-Fredholm elements in A modulo J .*

- i) If a_1a_2 and a_2a_1 are elements of J , then $a_1 + a_2$ is a B-Fredholm element in A modulo J .*
- ii) If $a_1a_2 = a_2a_1$, then a_1a_2 is a B-Fredholm element in A modulo J .*
- iii) If j is an element of J , then $a_1 + j$ is a B-Fredholm element in A modulo J .*

Proof. i) We have $\pi(a_1)\pi(a_2) = \pi(a_2)\pi(a_1) = 0$, From [14, Corollary 1], it follows that $\pi(a_1 + a_2) = \pi(a_1) + \pi(a_2)$ is Drazin invertible in A/J . So $a_1 + a_2$ is a B-Fredholm element in A .

ii) We have $\pi(a_1a_2) = \pi(a_1)\pi(a_2) = \pi(a_2)\pi(a_1)$. Similarly to [10, Proposition 2.6], it follows that $\pi(a_1a_2)$ is Drazin invertible in A/J . Hence a_1a_2 is a B-Fredholm element in A modulo J .

iii) If $i \in J$, then $\pi(a_1 + i) = \pi(a_1)$. So $a_1 + i$ is a B-Fredholm element in A modulo J .

The following proposition is well known in the case Banach algebras. Its inclusion here with a direct proof, in the case of unital rings, is for a seek of completeness.

PROPOSITION 2.5. *Let A be a ring with a unit e , and let $a \in A$. Then a is Drazin invertible in A if and only if there exists an integer $n \in \mathbb{N}^*$, such that $A = a^n A \oplus N(a^n)$, where $N(a^n) = \{x \in A \mid a^n x = 0\}$. In this case there exists two idempotents p, q such that $e = p + q, pq = qp = 0$ and $A = pA \oplus qA$.*

Proof. Assume that a is Drazin invertible in A . Then there exists $b \in A$ and $k \in \mathbb{N}$ such that $bab = b, ab = ba, a^k ba = a^k$. Without lose of generality, we can assume that $k = 1$. Let us show that $A = aA \oplus N(a)$. Since $aba = a$ then $aA = a^2A$. So if $x \in A$, then $ax = a^2t$, with $t \in A$. Hence $a(x - at) = 0$, and $x - at \in N(a)$. Therefore $x = at + (x - at)$. Moreover if $x \in aA \cap N(a)$, then $x = at, t \in A$. Hence $0 = bax = abat = at = x$. Thus $A = aA \oplus N(a)$.

Conversely, assume that $A = aA \oplus N(a)$. Then there exists $p \in aA, q \in N(a)$, such that $e = p + q$. Then $p = p^2 + qp$, and $p - p^2 = qp$. As $aA \cap N(a) = \{0\}$, then $p^2 = p$, and $qp = 0$. Similarly, we can show that $q^2 = q$, and $pq = 0$. Moreover, if $x \in A$, then $x = ex = px + qx$. As $pA \cap qA = \{0\}$, then $A = pA \oplus qA$. Thus, there exists $r \in A$, such that $a = pr$, hence $pa = p^2r = pr = a$. On the other side, we have $ap = a(e - q) = a$. Thus $ap = pa = a$. Similarly, we have $aq = qa = 0$. Since $A = aA \oplus N(a)$, then $aA = a^2A$. So there exists $b \in aA$, such that $p = ab$. Then $a = pa = aba$. We have $a(ba - p) = aba - ap = 0$, so $ba - p \in aA \cap N(a)$. Thus $ba = p$, and $ba = ab = p$. As $bab - b = (ba - e)b = qb \in aA \cap N(a)$, so $bab = b$. Finally we have $aba = a, bab = b, ab = ba$, and a is Drazin invertible in A .

THEOREM 2.6. *An element $a \in A$ is a B-Fredholm element of A modulo J if and only if there exists an integer $n \in \mathbb{N}^*$, an element $c \in A$ such that $a^n ca^n - a^n \in J$ and $e - a^n c - ca^n$ is a Fredholm element in A modulo J .*

Proof. Assume that a is B-Fredholm element in A modulo J . Then $\pi(a)$ is Drazin invertible in the quotient ring A/J . Hence there exists $b \in A$ and $k \in \mathbb{N}^*$

such that $\pi(b)\pi(a)\pi(b) = \pi(b)$, $\pi(a)\pi(b) = \pi(b)\pi(a)$, and $\pi(a)^{k+1}\pi(b) = \pi(a)^k$. So $\pi(a)^k\pi(b)^k = \pi(b)^k\pi(a)^k$, $\pi(b)^k\pi(a)^k\pi(b)^k = \pi(b)^k$, and $\pi(a)^k\pi(b)^k\pi(a)^k = \pi(a)^k[\pi(b)\pi(a)]^k = \pi(a)^k$. Let $c = b^k$, then $\pi(e) - \pi(a)^k\pi(c) - \pi(c)\pi(a)^k = \pi(e)$ is invertible in the quotient ring A/J .

Conversely suppose that there exists an integer $n \in \mathbb{N}$, and an element $c \in A$ such that $\pi(a)^n\pi(c)\pi(a)^n = \pi(a)^n$ and $\pi(e) - \pi(a)^n\pi(c) - \pi(c)\pi(a)^n$ is invertible in A/J . Let $t = \pi(e) - \pi(a)^n\pi(c) - \pi(c)\pi(a)^n$, $s = t^{-1}$ and let $L_{\pi(a)^n}$ be the left multiplication in A/J by $\pi(a)^n$, $Im(L_{\pi(a)^n})$ and $N(L_{\pi(a)^n})$ its image and kernel respectively. We have $\pi(a)^nt = -\pi(a)^{2n}\pi(c)$, and $\pi(a)^n = \pi(a)^n\pi(e) = \pi(a)^nts = -\pi(a)^{2n}\pi(c)s$. Hence $\pi(a)^nA/J = \pi(a)^{2n}A/J$, and so $Im(L_{\pi(a)^n}) = Im(L_{\pi(a)^{2n}})$.

Similarly we have $t\pi(a)^n = -\pi(c)\pi(a)^{2n}$, and $\pi(a)^n = \pi(e)\pi(a)^n = st\pi(a)^n = -s\pi(c)\pi(a)^{2n}$. Hence $N(L_{\pi(a)^n}) = N(L_{\pi(a)^{2n}})$. Then it can be easily seen that $A/J = Im(L_{\pi(a)^n}) \oplus N(L_{\pi(a)^n})$, where \oplus stands for direct sum. From Proposition 2.5, it follows that $\pi(a)$ is Drazin invertible in A/J , and a is a B-Fredholm element in A modulo J .

Let us recall that an operator $T \in L(X)$ has a generalized inverse if there is an operator $S \in L(X)$ such that $TST = T$. In this case S is called a generalized inverse of T . It is well known that T has a generalized inverse if and only if $R(T)$ and $N(T)$ are closed and complemented subspaces of X . In [12], S.R. Caradus has defined the following class of operators :

DEFINITION 2.7. $T \in L(X)$ is called a generalized Fredholm operator if T is relatively regular and there is a generalized inverse S of T such that $I - ST - TS$ is a Fredholm operator.

In [19] and [20], this class of operators had been studied and it is proved [20, Theorem 1.1] that an operator $T \in L(X)$ is a generalized Fredholm operator if and only if $T = Q \oplus F$, where Q is a finite dimensional nilpotent operator and F is a Fredholm operator. Then Theorem 2.6, encourages us to consider the following class of elements in a ring A , with a unit e .

DEFINITION 2.8. An element $a \in A$ is a generalized Fredholm element modulo J if there exists an element $b \in A$ such that $aba - a \in J$ and $e - ab - ba$ is a Fredholm element in A modulo J .

From Theorem 2.6, we obtain immediately the following characterization of B-Fredholm elements

THEOREM 2.9. *An element $a \in A$ is a B-Fredholm element in A modulo J if and only if there exists an integer $n \in \mathbb{N}^*$ such that a^n is a generalized Fredholm element in A modulo J .*

Let A be a complex Banach algebra, with unit e , $a \in A$, and let

$$\sigma_F(a) = \{\lambda \in \mathbb{C} \mid a - \lambda e \text{ is not a Fredholm element in } A \text{ modulo } J\},$$

and

$$\sigma_{BF}(a) = \{\lambda \in \mathbb{C} \mid a - \lambda e \text{ is not a B-Fredholm element in } A \text{ modulo } J\},$$

be respectively the Fredholm and the B-Fredholm spectrum of a . Then, we have the following result.

THEOREM 2.10. *Let A be a unital Banach algebra and $a \in A$. If f an analytic function in a neighborhood of the usual spectrum $\sigma(a)$ of a which is non-constant on any connected component of the usual spectrum $\sigma(a)$, of a , then $f(\sigma_{\mathbf{BF}}(a)) = \sigma_{\mathbf{BF}}(a)$.*

Proof. From Theorem 2.2 and Theorem 2.3, we know that the set of Fredholm (resp; B-Fredholm) elements in A is a regularity. Then the corollary is a direct consequence of [17, Theorem 1.4].

3 B-Fredholm elements in primitive Banach algebras

In this section, we will assume that A is a complex unital primitive Banach algebra, with unit e , and the ideal J is equal to its socle. Recall that an algebra is called primitive if $\{0\}$ is a primitive ideal of A . We will assume that the socle J of A is not reduced to $\{0\}$, so in this case and from [4], A possesses minimal idempotents. A minimal idempotent p of A , is a non zero idempotent p such that $pAp = \mathbb{C}e$. Recall also that it is well know that a primitive Banach algebra is a semi-prime algebra.

Let p is any minimal idempotent in A . For $a \in A$, consider the operator $\hat{a} : Ap \rightarrow Ap$, defined by $\hat{a}(y) = ay$, for all $y \in Ap$. We know from [4, F.2.6], that if a is a Fredholm element in A , then \hat{a} is a Fredholm operator on the Banach space Ap . However, the converse is in general false, as shown in [4, F.4.2].

An element $a \in A$ is said to be of finite rank if the operator \hat{a} is an operator of finite rank. We know from [4, Theorem F.2.4], that the socle of A is $\text{soc}(A) = \{x \in A \mid \hat{x} \text{ is of finite rank}\}$. The left regular representation of the Banach algebra A on the Banach space Ap is defined by $\mathfrak{L}_r : A \rightarrow L(Ap)$, such that $\mathfrak{L}_r(x) = \hat{x}$.

For more details about the notions from Fredholm theory in Banach algebras used here, we refer the reader to [4].

DEFINITION 3.1. [15, 2.1., p.283] Let I be an ideal in a Banach algebra A . A function $\tau : I \rightarrow \mathbb{C}$, is called a trace on I if :

- 1) $\tau(p) = 1$ if $p \in I$, is an idempotent that is $p^2 = p$, and p of rank one,
- 2) $\tau(a + b) = \tau(a) + \tau(b)$, for all $a, b \in I$,
- 3) $\tau(\alpha a) = \alpha \tau(a)$, for all $\alpha \in \mathbb{C}$ and $a \in I$,
- 4) $\tau(ab) = \tau(ba)$, for all $a \in I$ and $b \in A$.

From [2, Section 3], a trace function is defined on the socle by: $\tau(a) = \sum_{\lambda \in \sigma(a)} m(\lambda, a) \lambda$, for an element a of the socle of A , where $\sigma(a)$ is the spectrum of a , and $m(\lambda, a)$ is the algebraic multiplicity of λ for a .

DEFINITION 3.2. The index of a B-Fredholm element $a \in A$ is defined by:

$$\mathbf{i}(a) = \tau(aa_0 - a_0a) = \tau([a, a_0]),$$

where a_0 is a Drazin inverse of a modulo the socle J of A .

From [5, Theorem 2.3], the index of a B-Fredholm element $a \in A$ is well defined and is independant of the Drazin inverse a_0 of a modulo the ideal J .

DEFINITION 3.3. Let $a \in A$. Then a is called a B-Weyl element if it is a B-Fredholm element of index 0.

The following theorem gives, under more hypothesis, a decomposition result for B-Fredholm elements of index 0 in primitive Banach algebras, similar to [16, Theorem 3.1] and [16, Theorem 3.2], given for Fredholm elements of index 0, in semi-simple Banach algebras. It extends also a similar decomposition given for B-Fredholm operators acting on a Banach space in [9, Corollary 4.4].

THEOREM 3.4. *Let A be a unital primitive Banach algebra such that $\mathfrak{L}_r(A)$ is Drazin inverse closed in $L(Ap)$, and $\mathfrak{L}_r(\text{soc}(A)) = F_0(Ap)$, where $F_0(Ap)$ is the ideal of finite rank operators in $L(Ap)$. Then an element $a \in A$, is a B-Weyl element if and only if $a = b + c$ where b is a Drazin invertible element of A and c is an element of the socle J of A .*

Proof. Assume that a is a B-Fredholm element of index 0. Then from [5, Lemma 3.2], \hat{a} is a B-Fredholm operator of index 0. From [9, Corollary 4.4], we have $\hat{a} = S + F$, where S is a Drazin invertible operator and F is a finite rank operator. Since A satisfies $\mathfrak{L}_r(\text{soc}(A)) = F_0(Ap)$, there exist $c \in J$, such

that $F = \widehat{c}$. As $S = \widehat{a - c}$ is Drazin invertible in $L(Ap)$, and $\mathfrak{L}_r(A)$ is Drazin inverse closed in $L(Ap)$, then $\widehat{a - c}$ is Drazin invertible in $\mathfrak{L}_r(A)$. As the representation \mathfrak{L}_r is faithful [4, p. 30], then $a - c$ is Drazin invertible in A . Put $b = a - c$, then $a = b + c$, gives the desired decomposition.

Conversely if $a = b + c$ where b is a Drazin invertible element of A and c is an element of J , then from [5, Proposition 3.3], a is a B-Fredholm element of A of index 0.

EXAMPLE 3.5. From [4, Theorem F.4.3], if A , is a unital primitive C^* -algebra, then $\mathfrak{L}_r(A)$ is inverse closed in $L(Ap)$ and $\mathfrak{L}_r(\text{soc}(A)) = F_0(Ap)$. Thus from [18, Corollary 6], $\mathfrak{L}_r(A)$ is Drazin inverse closed in $L(Ap)$. Thus a primitive unital C^* -algebra satisfies the hypothesis of Theorem 3.4

The aim of the rest of this section, is to establish a connection between B-Fredholmness of an element a of A and of the B-Fredholmness of the operator $\mathfrak{L}_r(a) = \widehat{a}$.

THEOREM 3.6. *Let A be a primitive complex unital Banach algebra. If a is a B-Fredholm element of A modulo J , then the operator \widehat{a} is a B-Fredholm operator on the Banach space Ap .*

Proof. If a is a B-Fredholm element in A modulo J , then a is Drazin invertible in A modulo J . From [4, Theorem F.2.4], we know that J is exactly the set of elements x of A such that \widehat{x} is an operator of finite rank. Then \widehat{a} is a Drazin invertible operator modulo the ideal of finite rank on Ap . Thus from Theorem 1.1, $\widehat{a} : Ap \rightarrow Ap$ is a B-Fredholm operator.

However, the converse of Theorem 3.6 does not hold in general. To prove this, we use the same example as in [4, Example F.4.2]

EXAMPLE 3.7. Let T be the bilateral shift on the Hilbert space $l^2(\mathbb{Z})$. Consider the closed unital subalgebra of $L(l^2(\mathbb{Z}))$, generated by T and the ideal $K(l^2(\mathbb{Z}))$ of compact operators on $l^2(\mathbb{Z})$. It follows from [4, Example F.4.2] that A is a primitive Banach algebra and $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_A(T)$, the spectrum of T in A . Hence $0 \in \sigma_A(T)$, and it is not an isolated point of $\sigma_A(T)$. Therefore a is not a B-Fredholm element of A , otherwise and from [9, Remark A, iii)], if $\lambda \neq 0$ and $|\lambda|$ is small enough, then $T - \lambda I$ is a Fredholm operator. But this impossible since from [4, Example F.4.2], $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_{A/K(H)}(T + K(H))$, the spectrum of $T + K(H)$ in the Calkin algebra $L(H)/K(H)$.

In the following theorem, we give a necessary and sufficient condition, which ensures that the converse of Theorem 3.6 is true.

THEOREM 3.8. *let A be a primitive complex unital Banach algebra satisfying $\mathfrak{L}_r(\text{soc}(A)) = F_0(Ap)$. Then the following two conditions are equivalent:*

- i) For an element $a \in A$, if \widehat{a} is a B-Fredholm operator on the Banach space Ap , then a is a B-Fredholm element of A .*
- ii) Each element of the algebra $\mathfrak{L}_r(A)/F_0(Ap)$ which is Drazin invertible in the algebra $L(Ap)/F_0(Ap)$, is also Drazin invertible in $\mathfrak{L}_r(A)/F_0(Ap)$.*

Proof. It is clear that i) implies ii). So assume that for $a \in A$, \widehat{a} is a B-Fredholm operator on the Banach space Ap . From Theorem 1.1, \widehat{a} is Drazin invertible modulo the ideal of finite rank on Ap . Since we assume that ii) is true, then \widehat{a} is Drazin invertible in $\mathfrak{L}_r(A)/F_0(Ap)$. Thus there exist $b \in A$, such that $\widehat{ab} - \widehat{ba}, \widehat{bab} - \widehat{b}, \widehat{a^{n+1}b} - \widehat{a^n}$, are elements of $F_0(Ap)$. As the representation π is faithful, then $ab - ba, bab - b, a^{n+1}b - a^n$, are elements of J . Thus a is a B-Fredholm element of A .

EXAMPLE 3.9. If A is a unital primitive C^* -algebra, then from [4, Theorem F.4.3], we have $\mathfrak{L}_r(\text{soc}(A)) = F_0(Ap)$, where p is a self-adjoint minimal idempotent of A . Moreover if we assume that A is commutative, and for $a \in A$, the operator \widehat{a} is a B-Fredholm operator, then \widehat{a} is a B-Fredholm multiplier on the C^* - algebra Ap . If $\lambda \neq 0$ and $|\lambda|$ is small enough, then from [9, Remark A, iii)], $\widehat{a} - \lambda I$ is a Fredholm multiplier on the C^* - algebra Ap . From [1, Corollary 5.105], $\widehat{a} - \lambda I$ is of index is 0. Therefore and by [9, Remark A, iii)], \widehat{a} is also of index 0. Hence by Example 3.5 and Theorem 3.4, a is a B-Weyl element of A , and so a B-Fredholm element of A .

Similarly to Theorem 3.8, we have the following result, which we give without proof.

THEOREM 3.10. *let A be a primitive complex unital Banach algebra satisfying $\mathfrak{L}_r(\text{soc}(A)) = F_0(Ap)$. Then the following two conditions are equivalent:*

- i) For an element $a \in A$, if \widehat{a} is a Fredholm operator on the Banach space Ap , then a is a Fredholm element of A .*
- ii) Each element of the algebra $\mathfrak{L}_r(A)/F_0(Ap)$ which is invertible in the algebra $L(Ap)/F_0(Ap)$, is also invertible in $\mathfrak{L}_r(A)/F_0(Ap)$.*

EXAMPLE 3.11. From [4, Theorem F.4.3], if A , is a unital primitive C^* -algebra, then $\mathfrak{L}_r(A)$ is inverse closed in $L(Ap)$ and $\mathfrak{L}_r(\text{soc}(A)) = F_0(Ap)$. Thus an element a of A is a Fredholm element if and only if \widehat{a} is Fredholm operator.

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